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Radial velocity $= \frac{dr}{dt} = \dot{r}$

Transverse velocity $= r\frac{d\theta}{dt} = r\dot{\theta}$

Radial acceleration $= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \ddot{r} - r\dot{\theta}^2$

Transverse acceleration $= r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\cdot\frac{d\theta}{dt}$

$= r\ddot{\theta} + 2\dot{r}\dot{\theta}$

$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$

$= \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)$

Q.1 The velocities of a particle along and perpendicular to the radius vector from a fixed origin are λr^3 and $\mu\theta^2$. Show that the equation to the path is

$$\frac{\lambda}{\theta} = \frac{\mu}{2r^2} + c$$

and component accelerations are $2\lambda^2 r^3 - \mu^2\frac{\theta^4}{r}$ and $\lambda\mu r\theta^2 + 2\mu^2\frac{\theta^3}{r}$.

Proof Here, radial velocity $= \lambda r^2$

i.e. $\frac{dr}{dt} = \lambda r^3$ —— (1)

and transverse velocity $= \mu\theta^2$

$\Rightarrow r\frac{d\theta}{dt} = \mu\theta^2$ —— (2)

We eliminate t from (1) and (2).

$$\frac{dr}{r\,d\theta} = \frac{\lambda r^3}{\mu\theta^2} \Rightarrow \lambda\frac{d\theta}{\theta^2} = \mu\frac{dr}{r^3}$$

$$\Rightarrow \lambda\int\frac{d\theta}{\theta^2} = \mu\int\frac{dr}{r^3}$$

$$\Rightarrow -\frac{\lambda}{\theta} = -\frac{\mu}{2r^2} - c$$

$$\Rightarrow \boxed{\frac{\lambda}{\theta} = \frac{\mu}{2r^2} + c}$$ This is the required path.

Radial acceleration $= \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2$

$$= \frac{d}{dt}\left(\frac{dr}{dt}\right) - \frac{1}{r}\cdot\left(r\frac{d\theta}{dt}\right)^2$$

$$= \frac{d}{dt}(\lambda r^2) - \frac{1}{r}\cdot(\mu\theta^2)^2 \quad \text{[using (1) and (2)]}$$

$$= 2\lambda r\cdot\frac{dr}{dt} - \frac{\mu^2\theta^4}{r}$$

$$= 2\lambda r\cdot\lambda r^2 - \frac{\mu^2\theta^4}{r} \quad \text{[using (1)]}$$

$$= 2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r}$$

Transverse acceleration

$$= r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\cdot\frac{d\theta}{dt}$$

$$= r\frac{d}{dt}\left(\frac{d\theta}{dt}\right) + 2\lambda r^2\cdot\frac{\mu\theta^2}{r} = r\frac{d}{dt}\left(\frac{\mu\theta^2}{r}\right) + 2\lambda\mu\theta^2 r$$

$$= \mu r\left[\frac{1}{r}\cdot 2\theta\cdot\frac{d\theta}{dt} - \frac{\theta^2}{r^2}\frac{dr}{dt}\right] + 2\lambda\mu\theta^2 r$$

$$= \mu r\left[\frac{2\theta}{r}\cdot\frac{\mu\theta^2}{r} - \frac{\theta^2}{\cancel{r^2}}\cdot\lambda\cancel{r^2}\right] + 2\lambda\mu\theta^2 r$$

$$= \frac{2\mu^2\theta^3}{r} + \lambda\mu\theta^2 r$$

Q. 2 The velocities of a particle along and perpendicular to the radius from a fixed origin are λr and $\mu\theta$; find the path and show that the accelerations along and $\perp r$ to the radius vector are

$$\lambda^2 r - \frac{\mu^2\theta^2}{r} \text{ and } \mu\theta\left(\lambda + \frac{\mu}{r}\right).$$

Soln Given that radial velocity $\left(\frac{dr}{dt}\right) = \lambda r$ —(1)

and transverse velocity $\left(r\frac{d\theta}{dt}\right) = \mu\theta$ —(2)

To eliminate t, from (1) and (2), divide (1) by (2).

$$\frac{dr}{r\,d\theta} = \frac{\lambda r}{\mu\theta} \Rightarrow \mu\frac{dr}{r^2} = \lambda\frac{d\theta}{\theta}$$

$$\Rightarrow \mu\int \frac{dr}{r^2} = \lambda\int\frac{d\theta}{\theta} \quad \text{~~integrating~~},$$

$$\Rightarrow -\frac{\mu}{r} = \lambda\log\theta - K/r$$

$$\Rightarrow \log\theta = K - \frac{\mu}{\lambda r}$$

This is the required path.

Now, radial acceleration

$$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{d}{dt}\left(\frac{dr}{dt}\right) - \frac{1}{r}\left(r\frac{d\theta}{dt}\right)^2$$

$$= \frac{d}{dt}(\lambda r) - \frac{1}{r}\cdot(\mu\theta)^2 \quad [\text{using (1) and (2)}$$

$$= \lambda\frac{dr}{dt} - \frac{\mu^2\theta^2}{r} = \lambda\cdot\lambda r - \frac{\mu^2\theta^2}{r}$$

$\therefore$ radial acceleration $= \lambda^2 r - \dfrac{\mu^2\theta^2}{r}$
[Proved]

Transverse acceleration

$$= r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\cdot\frac{d\theta}{dt}$$

$$= r\frac{d}{dt}\left(\frac{d\theta}{dt}\right) + 2\lambda r\cdot\frac{\mu\theta}{r} \quad \text{[using (1) and (2)]}$$

$$= r\cdot\frac{d}{dt}\left(\frac{\mu\theta}{r}\right) + 2\lambda\mu\cancel{r}\frac{\theta}{\cancel{r}} = \mu r\frac{d}{dt}\left(\frac{\theta}{r}\right) + 2\lambda\mu\theta$$

$$= \mu r\left[\frac{1}{r}\frac{d\theta}{dt} - \frac{\theta}{r^2}\frac{dr}{dt}\right] + 2\lambda\mu\theta$$

$$= \mu r\left[\frac{1}{r}\cdot\frac{\mu\theta}{r} - \frac{\theta}{r^2}\cdot\lambda r\right] + 2\lambda\mu\theta$$

$$= \frac{\mu^2\theta}{r} - \lambda\mu\theta + 2\lambda\mu\theta$$

$$= \frac{\mu^2\theta}{r} + \lambda\mu\theta$$

$$= \mu\theta\left(\lambda + \frac{\mu}{r}\right).$$

Proved